

A note on the asymptotic expansion of the Lerch's transcendent

Xing Shi Cai^{*1} and José L. López^{†2}

¹*Department of Mathematics, Uppsala University, Uppsala, Sweden*

²*Departamento de Estadística, Matemáticas e Informática and INAMAT, Universidad Pública de Navarra, Pamplona, Spain*

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Abstract

In [7], the authors derived an asymptotic expansion of the Lerch's transcendent $\Phi(z, s, a)$ for large $|a|$, valid for $\Re a > 0$, $\Re s > 0$ and $z \in \mathbb{C} \setminus [1, \infty)$. In this paper we study the special case $z \geq 1$ not covered in [7], deriving a complete asymptotic expansion of the Lerch's transcendent $\Phi(z, s, a)$ for $z > 1$ and $\Re s > 0$ as $\Re a$ goes to infinity. We also show that when a is a positive integer, this expansion is convergent for $\Re z \geq 1$. As a corollary, we get a full asymptotic expansion for the sum $\sum_{n=1}^m z^n/n^s$ for fixed $z > 1$ as $m \rightarrow \infty$. Some numerical results show the accuracy of the approximation.

1 Introduction

The Lerch's transcendent (Hurwitz-Lerch zeta function) [2, §25.14(i)] is defined by means of the power series

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}, \quad a \neq 0, -1, -2, \dots,$$

on the domain $|z| < 1$ for any $s \in \mathbb{C}$ or $|z| \leq 1$ for $\Re s > 1$. For other values of the variables z, s, a , the function $\Phi(z, s, a)$ is defined by analytic continuation. In particular [7],

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - ze^{-x}} dx, \quad \Re a > 0, \quad z \in \mathbb{C} \setminus [1, \infty) \quad \text{and} \quad \Re s > 0. \quad (1)$$

^{*}This work is supported by the Knut and Alice Wallenberg Foundation and the *Ministerio de Economía y Competitividad* of the spanish government (MTM2017-83490-P). Email: xingshi.cai@math.uu.se

[†]Email: jl.lopez@unavarra.es

This function was investigated by Erdélyi [3, §1.11, eq. 1]. Although using a different notation $z = e^{2\pi ix}$, it was previously introduced by Lerch [12] and Lipschitz [13] in connection with Dirichlet's famous theorem on primes in arithmetic progression. If $x \in \mathbb{Z}$, the Hurwitz-Lerch zeta function reduces to the meromorphic Hurwitz zeta function $\zeta(s, a)$ [15, §2.3, eq. 2], with one single pole at $s = 1$. Moreover, $\zeta(s, 1)$ is nothing but the Riemann zeta function $\zeta(s)$.

Properties of the Lerch's transcendent have been studied by many authors. Among other results, we remark the following ones. Apostol obtains functional relations for $\Phi(e^{2\pi ix}, s, a)$ and gives an algorithm to compute $\Phi(e^{2\pi ix}, -n, a)$ for $n \in \mathbb{N}$ in terms of a certain kind of generalized Bernoulli polynomials [1]. The function $\Phi(e^{2\pi ix}, s, a)$ is used in [10] to generalize a certain asymptotic formula considered by Ramanujan. Asymptotic equalities for some weighted mean squares of $\Phi(e^{2\pi ix}, s, a)$ are given in [11]. Integral representations, as well as functional relations and expansions for $\Phi(z, s, a)$ may be found in [15, §2.5]. See Erdélyi et al. [6] for further properties. Here we want to remark the following two important properties of the Lerch's transcendent valid for $x, z, s \in \mathbb{C}$, $m \in \mathbb{N}$ [2, §25.14.3 and §25.14.4]:

$$\Phi(z, s, 1) = \frac{1}{z} \text{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^{n-1}}{n^s}, \quad |z| < 1,$$

where $\text{Li}_s(z)$ is the polylogarithm function [2, §25.12], and

$$\Phi(z, s, x) = z^m \Phi(z, s, x + m) + \sum_{n=0}^{m-1} \frac{z^n}{(x + n)^s}, \quad -x \notin \mathbb{N} \cup \{0\}.$$

In particular, when $x = 1$, the second property may be written in the form

$$\eta(z, s, m) := \sum_{n=1}^m \frac{z^n}{n^s} = \text{Li}_s(z) - z^{m+1} \Phi(z, s, m + 1). \quad (2)$$

The finite sum $\eta(z, s, m)$ for $z > 1$ is of interest in the study of *random records in full binary trees* by Janson [9]. In a full binary tree, each node has two child nodes and each level of the tree is full. Thus T_m , a full binary tree of height m , has $n = 2^{m+1} - 1$ nodes. In the random records model, each node in T_m is given a label chosen uniformly at random from the set $\{1, \dots, n\}$ without replacement. A node u is called a *record* when its label is the smallest among all the nodes on the path from u to the root node. Let $h(u)$ be the distance from u to the root. Let $X(T_m)$ be the (random) number of records in T_m . Then it is easy to see that the expectation of $X(T_m)$ is simply

$$\sum_{u \in T_n} \frac{1}{h(u) + 1} = \sum_{i=0}^{m+1} \frac{2^i}{i + 1} = \frac{\eta(2, 1, m + 1)}{2} = \frac{2^{m+1}}{m} + \mathcal{O}\left(\frac{2^m}{m^2}\right) = \frac{n}{m} + \mathcal{O}\left(\frac{n}{m^2}\right), \quad (3)$$

where the last step follows from elementary asymptotic computations [9, Remark 1.3]. A generalization of random records, called *random k -cuts*, requires a similar computation which

boils down to finding an asymptotic expansion of $\eta(2, b/k, m)$ for some $k \in \mathbb{N}$ and $1 \leq b \leq k$ as $m \rightarrow \infty$, see [4, §5.3.1]. Or more generally, asymptotic expansions of the function

$$F(z, s, a) := \Phi(z, s, a) - \frac{\text{Li}_s(z)}{z^a}, \quad (4)$$

that generalizes the function $\eta(z, s, m-1)$ defined in (2), from integer to complex values of the variable m : $\eta(z, s, m-1) = -z^m F(z, s, m)$. Complete asymptotic expansions, including error bounds, of $\Phi(z, s, a)$ for large a have been investigated in [7]. In particular, for $\Re a > 0$, $\Re s > 0$ and $z \in \mathbb{C} \setminus [1, \infty)$, we have that, for arbitrary $N \in \mathbb{N}$ [7, Theorem 1],

$$\Phi(z, s, a) = \sum_{n=0}^{N-1} c_n(z) \frac{(s)_n}{a^{n+s}} + \mathcal{O}(a^{-N-s}), \quad (5)$$

as $|a| \rightarrow \infty$. In this formula $(s)_n := s(s+1)\dots(s+n-1)$ is the Pochhammer symbol, $c_0(z) = (1-z)^{-1}$ and, for $n = 1, 2, 3, \dots$,

$$c_n(z) := \frac{(-1)^n \text{Li}_{-n}(z)}{n!}. \quad (6)$$

From the identities (2) and (5) we have that, for all $N \in \mathbb{N}$, $z \notin [1, \infty)$ and $\Re s > 0$ where expansion (5) is valid,

$$\eta(z, s, m-1) = \text{Li}_s(z) - \frac{z^m}{m^s} \left[\sum_{n=0}^{N-1} c_n(z) \frac{(s)_n}{m^n} + \mathcal{O}(m^{-N}) \right], \quad (7)$$

as $m \rightarrow \infty$. Unfortunately, expansion (5) has not been proved for $z \in [1, \infty)$, and then, in principle, the above expansion of $\eta(z, s, m-1)$ does not hold in the domain of the variable z where the approximation of $\eta(z, s, a)$ has a greater interest. Had the expansion (5) been proved for $z \geq 1$, the asymptotic computations in (3) would have become unnecessary and an arbitrarily precise approximation could be achieved automatically from (7).

However, to our surprise, it seems that the expansion (7) is still valid when $z > 1$. An argument that supports this claim is the following. On the one hand, assuming for the moment that (7) holds for $z = 2$, then

$$\eta(2, -1, m) = \sum_{n=1}^m n2^n = (m-1)2^{m+1} + 2 + \mathcal{O}(m^{-N}), \quad N \in \mathbb{N}.$$

On the other hand, using summation by parts [8, pp. 56], it is easy to see that

$$\eta(2, -1, m) = \sum_{n=1}^m n2^n = (m-1)2^{m+1} + 2.$$

Thus expansion (7) seems to be correct for $z = 2$. Numerical experiments further suggest that this is also true for other values of $z > 1$.

Then, the purpose of this paper is to show that expansion (7) holds for $z > 1$. More generally, to derive an expansion of $F(z, s, a)$ for large $\Re a$ with $\Re s > 0$ and $z \geq 1$.

2 An expansion of $\Phi(z, s, a)$ for large $\Re a$ and $z \geq 1$

The main result of the paper is given in Theorem 1 below. In order to formulate Theorem 1, we need to consider the function

$$f(z, x, a) := \frac{1 - (ze^{-x})^{1-a}}{1 - ze^{-x}}, \quad (8)$$

and its Taylor coefficients $C_n(z, a)$ at $x = 0$. We also need the two following Lemmas.

Lemma 1. For $a, z \in \mathbb{C}$ and $n = 0, 1, 2, \dots$,

$$C_n(z, a) := c_n(z) - z^{1-a} p_n(z, a), \quad p_n(z, a) := \sum_{k=0}^n \frac{c_{n-k}(z)}{k!} (a-1)^k, \quad (9)$$

where, for $z \neq 1$, the coefficients $c_n(z)$ have been introduced in (6): $c_0(z) = (1-z)^{-1}$ and, for $n = 1, 2, 3, \dots$,

$$c_n(z) := \frac{(-1)^n \text{Li}_{-n}(z)}{n!}. \quad (10)$$

For $n = 1, 2, 3, \dots$, the coefficients $c_n(z)$ may be computed recursively in the form $nc_n(z) = -zc'_{n-1}(z)$. Observe that $p_n(z, a)$ are polynomials of degree n in the variable a .

For $z = 1$, the definition (9) must be understood in the limit sense. More precisely, $C_0(1, a) = 1 - a$ and, for $n = 1, 2, 3, \dots$,

$$C_n(1, a) = \frac{B_{n+1} - B_{n+1}(a-1) - (n+1)(a-1)^n}{(n+1)!}, \quad (11)$$

where B_n are the Bernoulli numbers and $B_n(a)$ the Bernoulli polynomials [5, §24.2].

Proof. Formulas (9)-(10) may be derived by combining the Taylor expansions at $x = 0$ of $e^{(a-1)x}$ and $(1 - ze^{-x})^{-1}$. Formula (11) follows from the generating function of the Bernoulli polynomials $B_n(a)$ [8, Eq. 7.81],

$$\frac{e^{(a-1)x}}{1 - e^{-x}} = \sum_{n=0}^{\infty} B_n(a) \frac{x^{n-1}}{n!},$$

the definition of the Bernoulli numbers $B_n = B_n(0)$, and the Taylor expansion of $1 - e^{-x(1-a)}$ at $x = 0$. The derivation of the recursion $nc_n(z) = -zc'_{n-1}(z)$ is straightforward (see [7]). \square

Apart from the explicit form of the coefficients $C_n(z, a)$ given above, we may compute them by means of the recurrence relation given in the following lemma.

Lemma 2. For $z \neq 1$ we have that $C_0(z, a) = \frac{1 - z^{1-a}}{1 - z}$ and, for $n = 1, 2, 3, \dots$,

$$C_n(z, a) = \frac{z}{1 - z} \left[\sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n-k)!} C_k(z, a) - \frac{(a-1)^n}{n! z^a} \right].$$

For $z = 1$ we have $C_0(1, a) = 1 - a$, and, for $n = 1, 2, 3, \dots$,

$$C_n(1, a) = -\frac{(a-1)^{n+1}}{(n+1)!} - \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{(n+1-k)!} C_k(1, a).$$

Proof. Replace

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \quad \text{and} \quad f(z, x, a) = \sum_{k=0}^{\infty} C_k(z, a) x^k$$

into the identity $[1 - ze^{-x}]f(z, x, a) = 1 - (ze^{-x})^{1-a}$ and equate the coefficients of equal powers of x . \square

Theorem 1. For fixed $N \in \mathbb{N}$, $\Re a > 1$ and $\Re s > 0$,

$$F(z, s, a) := \Phi(z, s, a) - \frac{\text{Li}_s(z)}{z^a} = \sum_{n=0}^{N-1} C_n(z, a) \frac{(s)_n}{a^{n+s}} + R_N(z, s, a), \quad (12)$$

with $C_n(z, a)$ given in the previous lemmas and, for $z > 1$,

$$R_N(z, s, a) = \mathcal{O}((\Re a)^{1-N-s} + az^{-\Re a}), \quad \Re a \rightarrow \infty. \quad (13)$$

This means that expansion (12) has an asymptotic character for large $\Re a$ when $z > 1$. Moreover, expansion (12) is convergent for $a = m = 2, 3, 4, \dots$ and $z \geq 1$; i.e., $R_N(z, s, m) \rightarrow 0$ as $N \rightarrow \infty$ and

$$F(z, s, m) := -\frac{1}{z^m} \sum_{k=1}^{m-1} \frac{z^k}{k^s} = \sum_{n=0}^{\infty} C_n(z, m) \frac{(s)_n}{m^{n+s}}, \quad z \geq 1. \quad (14)$$

Remark 1. Note that, both $\Phi(z, s, a)$ and $\text{Li}_s(z)$ are multivalued functions of z . In our analysis and numeric experiments we always choose the principle branch for the variable z . Thus (12) can be transformed into an asymptotic expansion of $\Phi(z, s, a)$ without ambiguity as follows

$$\Phi(z, s, a) = \frac{\text{Li}_s(z)}{z^a} + \sum_{n=0}^{N-1} C_n(z, a) \frac{(s)_n}{a^{n+s}} + R_N(z, s, a).$$

Proof. Using the integral representation (2) of $\Phi(z, s, a)$ given in [7] and the integral representation [2, §25.12.11] of the polylogarithm,

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - z} dx,$$

we find the following integral representation of the function $F(z, s, a)$ defined in (4):

$$F(z, s, a) := \Phi(z, s, a) - \frac{\text{Li}_s(z)}{z^a} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} f(z, x, a) dx, \quad (15)$$

valid for $\Re a > 0$, $\Re s > 0$ and $z \in \mathbb{C}$, with $f(z, x, a)$ given in (8). In principle, the left hand side of (15) is an analytic function of z in $\{z : |z| < 1; |\arg z| < \pi\}$. Then, the right hand side of this equation defines the analytic continuation of $F(z, s, a)$ in the variable z to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$.

The function $f(z, x, a)$ has the following Taylor expansion at $x = 0$:

$$f(z, x, a) = \sum_{k=0}^{n-1} C_k(z, a) x^k + r_n(z, x, a), \quad (16)$$

where the coefficients $C_n(z, a)$ are given in (9)-(10)-(11). Replacing the function $f(z, x, a)$ in the integral (15) by its Taylor expansion (16), and interchanging sum and integral, we obtain (12) with

$$R_n(z, s, a) := \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} r_n(z, x, a) dx. \quad (17)$$

Using the Cauchy integral formula for the remainder $r_n(z, x, a)$ we find

$$r_n(z, x, a) = \frac{x^n}{2\pi i} \int_C \frac{f(z, w, a)}{(w - x)w^n} dw, \quad f(z, w, a) := \frac{1 - (ze^{-w})^{1-a}}{1 - ze^{-w}}, \quad (18)$$

where we choose C to be a closed loop that encircles the points $w = 0$ and $w = x$, it is traversed in the positive direction, and is inside the region $U := \{w \in \mathbb{C}, \Re w > -W, |\Im w| < W\}$, for some arbitrary but fixed $W \in (0, 2\pi)$. (U is an infinity rectangular region around the positive real line $[0, \infty)$ of width $2W$ (see [7] for further details). Figure 1 gives an example of U and C .

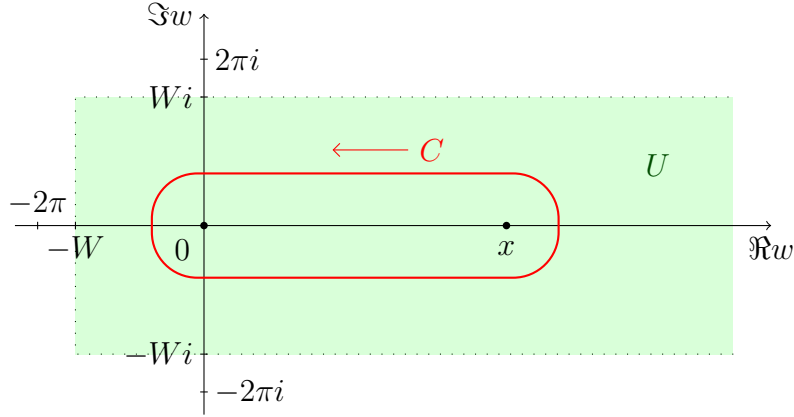


Figure 1: The integration loop C and the region U

The function $f(z, w, a)$ is continuous in the variable w for $w \in U$. The singularities of this function are $w = \log z + 2i\pi n$, $n \in \mathbb{Z} \setminus \{0\}$ and are located outside U . Define $b := \lfloor \Re a \rfloor$

and $\beta := a - b$ and write

$$\begin{aligned} f(z, w, a) &= \frac{1 - (ze^{-w})^{1-a}}{1 - ze^{-w}} = \frac{1 - (ze^{-w})^{-\beta} + (ze^{-w})^{-\beta} - (ze^{-w})^{1-b-\beta}}{1 - ze^{-w}} \\ &= \frac{1 - (ze^{-w})^{-\beta}}{1 - ze^{-w}} + (ze^{-w})^{-\beta} \frac{1 - (ze^{-w})^{1-b}}{1 - ze^{-w}}. \end{aligned} \quad (19)$$

We have that

$$\frac{1 - (ze^{-w})^{1-b}}{1 - ze^{-w}} = \sum_{k=0}^{b-2} \frac{(ze^{-w})^{-k} - (ze^{-w})^{-k-1}}{1 - ze^{-w}} = - \sum_{k=1}^{b-1} \left(\frac{e^w}{z} \right)^k. \quad (20)$$

The following bounds are valid for $w \in U$ and a certain constant $M_0 > 0$ independent of b and $\Re w$:

$$\left\{ \begin{array}{ll} \left| \frac{1 - (ze^{-w})^{-\beta}}{1 - ze^{-w}} \right|, & |(ze^{-w})^{-\beta}| \leq M_0 e^{\Re \beta \Re w}, \quad \text{for } 0 \leq \Re \beta < 1, \\ \left| \sum_{k=1}^{b-1} \left(\frac{e^w}{z} \right)^k \right| & \leq (b-1) \frac{e^{\Re w}}{z}, \quad \text{for } \Re w \leq \log z, \\ \left| \sum_{k=1}^{b-1} \left(\frac{e^w}{z} \right)^k \right| & \leq (b-1) \left(\frac{e^{\Re w}}{z} \right)^{b-1}, \quad \text{for } \Re w \geq \log z. \end{array} \right. \quad (21)$$

Therefore, from (19), (20) and (21) we have that, for any $w \in U$ and $\Re a > 1$,

$$|f(z, w, a)| \leq M|a|[e^{\Re w} + z^{1-\Re a} e^{(\Re a-1)\Re w}],$$

for a certain constant $M > 0$ independent of $\Re a$ and $\Re w$. The path C in (18) may be chosen in such a way that $\Re w \leq x + 1/\Re a$. Then, from (18), we find the following bound for the remainder $r_n(z, x, a)$:

$$|r_n(z, x, a)| \leq M_n |a| x^n [e^x + z^{1-\Re a} e^{x(\Re a-1)}], \quad \text{for } x \geq 0,$$

where M_n is a certain positive constant that depends on the geometry of the path C chosen in (18) and $\Im a$ but not on $\Re a$. Then, from (17),

$$|R_n(z, s, a)| \leq \frac{M_n |a|}{|\Gamma(s)|} \int_0^\infty x^{n+s-1} [e^{(1-\Re a)x} + z^{1-\Re a} e^{-x}] dx = \mathcal{O}((\Re a)^{1-n-s} + a z^{1-\Re a}).$$

This proves (13) and the asymptotic character of the expansion (12) for large $\Re a$ when $z > 1$.

Finally, we prove formula (14). When $a = m \in \{2, 3, 4, \dots\}$, the Taylor coefficients $C_n(z, m)$ of $f(z, x, m)$ at $x = 0$ are given in (9)-(10)-(11) with $a = m$. But we may derive a simpler formula for $C_n(z, m)$. We have that

$$f(z, x, m) := \frac{1 - (ze^{-x})^{1-m}}{1 - ze^{-x}} = \sum_{k=0}^{m-2} \frac{(ze^{-x})^{-k} - (ze^{-x})^{-k-1}}{1 - ze^{-x}} = - \sum_{k=1}^{m-1} \frac{e^{kx}}{z^k}.$$

Then, the Taylor coefficients $C_n(z, m)$ of $f(z, x, m)$ at $x = 0$ are the sum of the Taylor coefficients of $z^{-k}e^{kx}$, that is,

$$C_n(z, m) = -\frac{1}{n!} \sum_{k=1}^{m-1} \frac{k^n}{z^k} = \frac{1}{n!} [z^{-m}\Phi(z^{-1}, -n, m) - \text{Li}_{-n}(z^{-1})].$$

Since

$$|C_n(z, m)| = \frac{1}{n!} \sum_{k=1}^{m-1} \frac{k^n}{z^k} \leq \frac{(m-1)^n}{n!} \sum_{k=1}^{m-1} \frac{1}{z^k} \leq \frac{(m-1)^{n+1}}{n!},$$

we have that

$$\sum_{n=0}^{\infty} |C_n(z, m)| \int_0^{\infty} x^{n+\Re s-1} e^{-mx} dx \leq \frac{m-1}{m^{\Re s}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\Re s)}{n!} \left(\frac{m-1}{m}\right)^n < \infty.$$

Therefore, when $a = m \geq 2$, we may replace $f(z, x, m)$ into the integral (15) by its Taylor expansion (16) and interchange sum and integral. This proves (14). \square

Corollary 1. For fixed $N \in \mathbb{N}$, $z > 1$, $m \in \mathbb{N}$ and $\Re s > 0$,

$$\eta(z, s, m-1) = -\frac{z^m}{m^s} \left[\sum_{n=0}^{N-1} [c_n(z) + z^{1-m}p_n(z, m)] \frac{(s)_n}{m^n} + \mathcal{O}(m^{1+s-N} + m^{s+1}z^{-m}) \right],$$

as $m \rightarrow \infty$. Moreover, for $z \geq 1$,

$$\eta(z, s, m-1) = -\frac{z^m}{m^s} \sum_{n=0}^{\infty} [c_n(z) + z^{1-m}p_n(z, m)] \frac{(s)_n}{m^n}.$$

3 Final remarks and numeric experiments

Remark 2. The integral representation (1) of $\Phi(z, s, a)$ is not valid for $z \in [1, \infty)$ because of the pole of the integrand at $x = \log z$. This pole is removed by the subtraction of the function $x^{s-1}(e^x - z)^{-1}$ to the integrand. We obtain in this way the integral representation (15) of the function $F(z, s, a) := \Phi(z, s, a) - z^{-a}\text{Li}_s(z)$, free of the pole $x = \log z$ and valid for $z \in \mathbb{C} \setminus (-\infty, 0]$.

Remark 3. Since $f(z, x, 0) \equiv 1$, we have $C_0(z, 0) = 1$ and $C_n(z, 0) \equiv 0$ for $n = 1, 2, 3, \dots$. Thus by (9), for all $n \in \mathbb{N}$ and $z \neq 1$,

$$c_n(z) = \frac{z}{1-z} \sum_{k=0}^{n-1} \frac{(-1)^{n-k} c_k(z)}{(n-k)!},$$

which is equivalent to

$$\text{Li}_{-n}(z) = \frac{z}{(1-z)^2} + \frac{z}{1-z} \sum_{k=1}^{n-1} \binom{n}{k} \text{Li}_{-k}(z).$$

Remark 4. Observe that the terms of the expansion (12) are not a pure Poincaré expansion in the asymptotic sequence a^{-k-s} , as the coefficients $C_k(z, a) = c_k(z) + z^{1-a}p_k(z, a)$, $z \geq 1$, depend on a ($p_k(z, a)$ is a polynomial of degree k in a). For $z > 1$ these coefficients are separable and then we may write the expansion (12) in the form

$$\begin{aligned} F(z, s, a) &= \sum_{k=0}^{n-1} c_k(z) \frac{(s)_k}{a^{k+s}} + z^{1-a} \sum_{k=0}^{n-1} p_k(z, a) \frac{(s)_k}{a^{k+s}} + R_n(z, s, a) \\ &= \sum_{k=0}^{n-1} c_k(z) \frac{(s)_k}{a^{k+s}} + \mathcal{O}((\Re a)^{1-n-s} + az^{1-\Re a}). \end{aligned}$$

The first expansion is the expansion (12) of $\Phi(z, a, s)$, valid for $z \notin [1, \infty)$. The second one is an exponentially small correction; when z is very large, it is a small correction, but when z is close to 1, it is not negligible.

Remark 5. Using the summation by parts formula [14, §2.10.9], we have that

$$\begin{aligned} \eta(z, s, m) &= \frac{z^{m+1}}{z-1} \frac{1}{(m+1)^s} - \frac{z}{z-1} + \frac{z}{z-1} \sum_{k=1}^m \left(1 - (1+k^{-1})^{-s}\right) \frac{z^k}{k^s} \\ &= \frac{z^{m+1}}{z-1} \frac{1}{(m+1)^s} - \frac{z}{z-1} + \frac{z}{z-1} \sum_{k=1}^m \left(\sum_{n=1}^{\infty} \frac{(s)_n (-1)^{n-1}}{n!} \frac{1}{k^n} \right) \frac{z^k}{k^s} \\ &= \frac{z^{m+1}}{z-1} \frac{1}{(m+1)^s} - \frac{z}{z-1} + \frac{z}{z-1} \sum_{n=1}^{\infty} \frac{(s)_n (-1)^{n-1}}{n!} \eta(z, s+n, m). \end{aligned}$$

Thus, expansion (7) could also be proved by induction on N using the above identity.

Remark 6. Note that, for $z = 1$,

$$-F(1, s, m) = \eta(1, s, m-1) = \sum_{n=1}^{m-1} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=m}^{\infty} \frac{1}{n^s} = \zeta(s) - \zeta(s, m),$$

where $\zeta(s)$ denotes the *Riemann zeta function* [2, §25.2] and $\zeta(s, m)$ denotes the *Hurwitz zeta function* [2, §25.11]. Thus (14) gives a new series representation of $\zeta(s, m)$ for $m \in \mathbb{N}$, as

$$\begin{aligned} \zeta(s, m) &= \zeta(s) + \sum_{k=0}^{\infty} C_k(1, m) \frac{(s)_k}{m^{k+s}} \\ &= \zeta(s) + \frac{1}{m^s} \left(1 - m + \sum_{k=1}^{\infty} \frac{B_{k+1} - B_{k+1}(m-1) - (k+1)(m-1)^k (s)_k}{(k+1)!} \frac{(s)_k}{m^k} \right) \\ &= \zeta(s) + \frac{1}{m^s} \left(2 - m - m^s + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{(s)_{2k-1}}{m^{2k-1}} - \sum_{k=1}^{\infty} \frac{B_{k+1}(m-1) (s)_k}{(k+1)!} \frac{(s)_k}{m^k} \right). \end{aligned}$$

On the other hand, using the *Euler-Maclaurin's summation formula* [8, §9.5], or the asymptotic expansion of $\zeta(s, m)$ in [2, Eq. 25.11.43], we have, for $s > 1$,

$$F(1, s, m) = -\zeta(s) + \frac{m^{1-s}}{s-1} + \frac{m^{-s}}{2} + \sum_{k=1}^n \frac{B_{2k}}{(2k)!} \frac{(s)_{2k-1}}{m^{2k+s-1}} + \hat{R}_n(s, m),$$

with

$$\left| \hat{R}_n(s, m) \right| \leq \frac{|B_{2n+2}|}{(2n+2)!} \frac{|(s)_{2n+1}|}{m^{2n+s+1}}.$$

The following tables and pictures show some numerical experiments about the accuracy of the approximations given in Theorem 1. In the tables we compute the absolute value of the relative error $\bar{R}_n(z, s, a)$ in the approximation (12), defined in the form

$$\bar{R}_n(z, s, a) := 1 - \frac{\sum_{k=0}^{n-1} C_k(z, a) \frac{(s)_k}{a^{k+s}}}{F(z, s, a)}.$$

In Table 1 and Figure 2 we evaluate the Lerch's transcendent, the Polylogarithm and all the approximations with the symbolic manipulator *Wolfram Mathematica 10.4*.

$z = 2, s = 1$				$z = 5, s = 2$			
n	$a = 5$	$a = 10$	$a = 20$	n	$a = 5$	$a = 10$	$a = 20$
5	7.87e-2	2.22e-2	6.21e-4	5	8.36e-2	2.57e-3	5.87e-5
10	2.13e-2	7.55e-3	7.36e-5	10	2.82e-2	2.89e-4	1.21e-7
15	6.69e-3	3.68e-3	3.24e-5	15	1.13e-2	1.23e-4	2.66e-9

$z = 2, s = 2$				$z = 5, s = 3$			
n	$a = 10 + i$	$a = 30 + i$	$a = 50 + i$	n	$a = 10 + i$	$a = 30 + i$	$a = 50 + i$
5	1.60e-1	3.67e-4	2.41e-5	5	9.59e-3	2.52e-5	1.91e-6
10	9.14e-2	1.11e-5	3.32e-8	10	2.37e-3	1.04e-8	5.78e-11
15	5.92e-2	3.62e-6	4.75e-10	15	1.43e-3	3.47e-11	1.35e-14

Table 1: The relative error in the approximation (12) for Lerch's transcendent

4 Acknowledgements

The referee is acknowledged for his/her useful comments and improving suggestions. In particular, for suggesting the following alternative approach to derive our main result. Integral (1) can be written in the form

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{x - \log z} dx + \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \left[\frac{1}{1 - ze^{-x}} - \frac{1}{x - \log z} \right] dx.$$

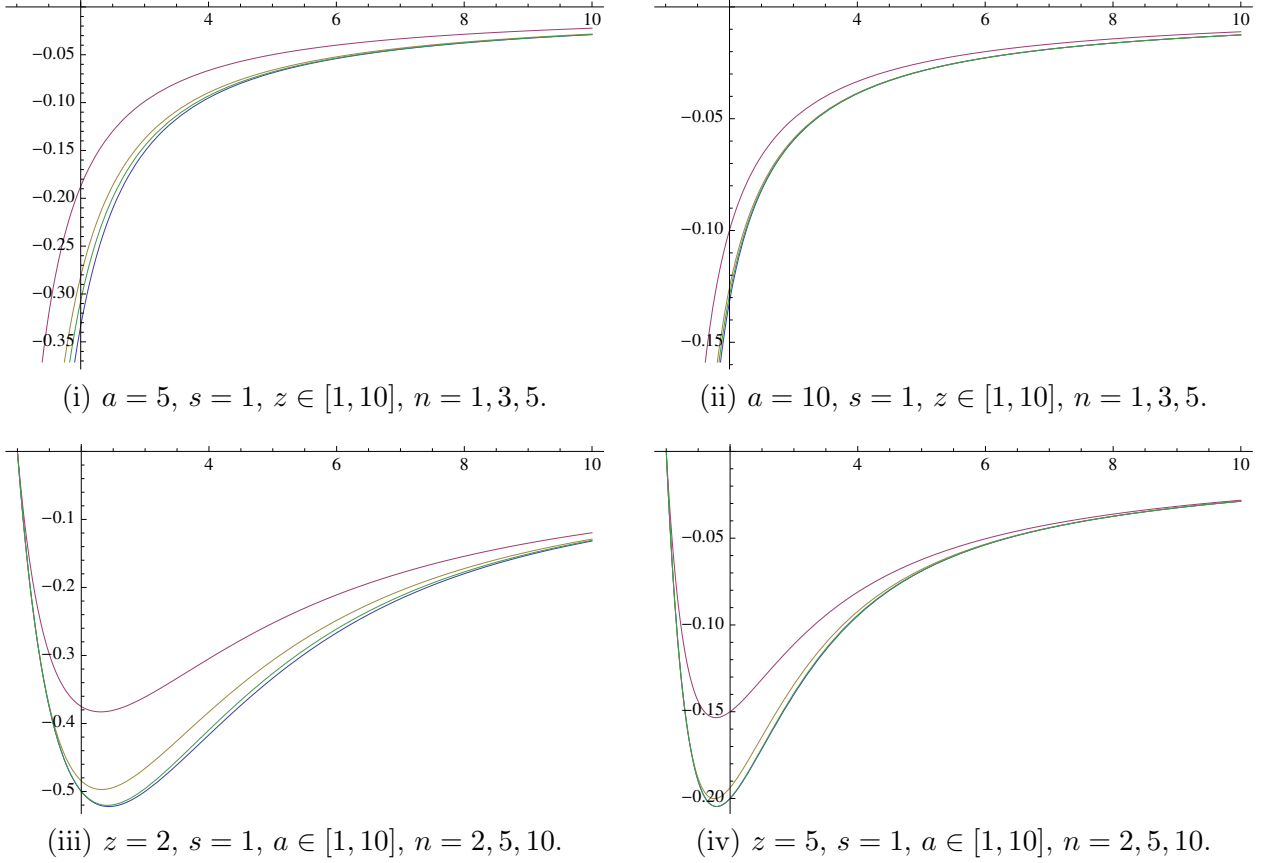


Figure 2: The blue line is the graphic of the function $F(z, s, a)$, whereas the red, gold and green functions represent the right hand side of (12) for increasing values of the approximation order n .

An asymptotic expansion of the second integral for large a may be derived by using Watson's lemma. The first integral can be written as an incomplete gamma function and takes over the role of the polylogarithm in our approach.

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